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ON HODGE THEORY OF SINGULAR PLANE CURVES

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Abstract. The dimensions of the graded quotients of the cohomology of a plane curve complement $U = \mathbb{P}^2 \setminus C$ with respect to the Hodge filtration are described in terms of simple geometrical invariants. The case of curves with ordinary singularities is discussed in detail. We also give a precise numerical estimate for the difference between the Hodge filtration and the pole order filtration on $H^2(U, \mathbb{C})$.

1. Introduction

The Hodge theory of the complement of projective hypersurfaces have received a lot of attention, see for instance Griffiths [10] in the smooth case, Dimca-Saito [5] and Sernesi [12] in the singular case. In this paper we consider the case of plane curves and continue the study initiated by Dimca-Sticlaru [7] in the nodal case and the author [1] in the case of plane curves with ordinary singularities of multiplicity up to 3.

In the second section we compute the Hodge-Deligne polynomial of a plane curve $C$, the irreducible case in Proposition 2.1 and the reducible case in Proposition 2.2. Using this we determine the Hodge-Deligne polynomial of $U = \mathbb{P}^2 \setminus C$ and then we deduce in Theorem 2.7 the dimensions of the graded quotients of $H^2(U)$ with respect to the Hodge filtration.

In section three we consider the case of arrangements of curves having ordinary singularities and intersecting transversely at smooth points and obtain a formula in Theorem 3.1 generalizing the formulas obtained in [7] and in [1] (for this type of curves). In fact, the results in [1] show that this formula holds in the more general case of plane curves with ordinary singularities of multiplicity up to 3 (without assuming transverse intersection).

In the forth section we show that the case of plane curves with ordinary singularities of multiplicity up to 4 (without assuming transverse intersection) is definitely more complicated and the formula in Theorem 3.1 has to be replaced by the formula in Theorem 4.1 containing a correction term coming from triple points on one component through which another component of $C$ passes.

In the final section we give some applications, we hope of general interest, expressing the difference between the Hodge filtration and the pole order filtration on $H^2(U, \mathbb{C})$ in terms of numerical invariants easy to compute in given situations, see Theorem 5.1 and its corollaries. One example involving a free divisor concludes this note.

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2. Hodge Theory of Plane Curve Complements

For the general theory of mixed Hodge structures we refer to [2] and [14]. Recall the definition of the Hodge-Deligne polynomial of a quasi-projective complex variety $X$

\[ P(X)(u, v) = \sum_{p,q} E^{p,q}(X) u^p v^q \]

where $E^{p,q}(X) = \sum_s (-1)^s h^{p,q}(H^s_c(X))$, with $h^{p,q}(H^s_c(X)) = \dim \text{Gr}^p_{F^p} \text{Gr}^W_{p+q} H^s_c(X, \mathbb{C})$, the mixed Hodge numbers of $H^s_c(X)$.

This polynomial is additive with respect to constructible partitions, i.e. $P(X) = P(X \setminus Y) + P(Y)$ for a closed subvariety $Y$ of $X$. In this section we determine $P(C)$ for a (reduced) plane curve $C$.

Suppose first that the curve $C$ is irreducible, of degree $N$. Denote by $a_k$, $k = 1, ..., p$ the singular points of $C$, and let $r(C, a_k)$ be the number of irreducible branches of the germ $(C, a_k)$. Let $\nu : \tilde{C} \to C$ be the normalization mapping. Using the normalization map $\nu$ and the additivity of the Hodge-Deligne polynomial, it follows that,

\[ P(C) = P(C \setminus (C)_{\text{sing}}) + P((C)_{\text{sing}}) = P(\tilde{C} \setminus (\bigcup_k \nu^{-1}(a_k))) + p = \]

\[ = P(\tilde{C}) - \sum_k P(\nu^{-1}(a_k)) + p = uv - gu - gv + 1 - \sum_k (r(C, a_k) - 1). \]

Indeed, it is known that for the smooth curve $\tilde{C}$, the genus $g = g(\tilde{C})$ is exactly the Hodge number $h^{1,0}(\tilde{C}) = h^{0,1}(\tilde{C})$. Moreover, it is known that one has the formula

\[ g = (N - 1)(N - 2)/2 - \sum_k \delta(C, a_k), \]

relating the genus, the degree and the local singularities of $C$, and the $\delta$-invariants can be computed using the formula

\[ 2\delta(C, a_k) = \mu(C, a_k) + r(C, a_k) - 1, \]

where $\mu(C, a_k)$ is the Milnor number of the singularity $(C, a_k)$. For both formulas above, see Milnor, p. 85. This proves the following result.

**Proposition 2.1.** With the above notation and assumptions, we have the following for an irreducible plane curve $C \subset \mathbb{P}^2$.

(i) The Hodge-Deligne polynomial of $C$ is given by

\[ P(C)(u, v) = uv - gu - gv + 1 - \sum_k (r(C, a_k) - 1), \]

with $g$ given by the formula (2.1).

(ii) $H^0(C) = \mathbb{C}$ is pure of type $(0, 0)$.

(iii) $H^2(C) = \mathbb{C}$ is pure of type $(1, 1)$.

(iv) The mixed Hodge numbers of the MHS on $H^1(C)$ are given by

\[ h^{0,0}(H^1(C)) = \sum_k (r(C, a_k) - 1), \quad h^{1,0}(H^1(C)) = h^{0,1}(H^1(C)) = g. \]
In particular, one has the following formulas for the first Betti number of $C$.

$$b_1(C) = \sum_k (r(C, a_k) - 1) + 2g = (N - 1)(N - 2) - \sum_k \mu(C, a_k).$$

Now we consider the case of a curve $C$ having several irreducible components. More precisely, let $C = \bigcup_{j=1}^r C_j$ be the decomposition of $C$ as a union of irreducible components $C_j$, let $\nu_j : \tilde{C}_j \to C_j$ be the normalization mappings and set $g_j = g(\tilde{C}_j)$. Suppose that the curve $C_j$ has degree $N_j$, denote by $a_k^j$ for $k = 1, \ldots, p_j$ be the singular points of $C_j$ and let $r(C_j, a_k^j)$ be the number of branches of the germ $(C_j, a_k^j)$. Then the formulas (2.1) and (2.2) can be applied to each irreducible curve $C_j$, as well as Proposition 2.1.

Let $A$ be the union of the singular sets of the curves $C_j$. Let $B$ be the set of points in $C$ sitting on at least two distinct components $C_i$ and $C_j$. For $b \in B$, let $n(b)$ be the number of irreducible components $C_j$ passing through $b$. By definition, $n(b) \geq 2$. Moreover, note that the sets $A$ and $B$ are not disjoint in general, and their union is precisely the singular set of $C$.

Using the additivity of Hodge-Deligne polynomials we get

$$P(C) = P(C_1 \cup \cdots \cup C_r) = \sum_{j=1}^r P(C_j) + (-1)^{l-1} \sum_{0 \leq i_1 < \cdots < i_l \leq r} P(C_{i_1} \cap \cdots \cap C_{i_l}).$$

The first sum is easy to determine using Proposition 2.1.

$$\sum_{j=1}^r P(C_j) = ruv - \left( \sum_{j=1}^r g_j \right) u - \left( \sum_{j=1}^r g_j \right) v + r - \sum_{j,k} (r(C_j, a_k^j) - 1).$$

Consider now the alternated sum, where $l \geq 2$. The only points of $C$ that give a contribution to this sum are the points in $B$. Now, for a point $b \in B$, its contribution to the alternated sum is clearly given by

$$c(b) = -\frac{n(b)}{2} + \frac{n(b)}{3} - \cdots - (-1)^{n(b)-1} \frac{n(b)}{n(b)} = -n(b) + 1.$$

**Proposition 2.2.** With the above notation and assumptions, we have the following for a reducible plane curve $C = \bigcup_{j=1}^r C_j$.

(i) The Hodge-Deligne polynomial of $C$ is given by

$$P(C)(u, v) = ruv - \left( \sum_{j=1}^r g_j \right) u - \left( \sum_{j=1}^r g_j \right) v + r - \sum_{j,k} (r(C_j, a_k^j) - 1) - \sum_{b \in B} (n(b) - 1).$$

with $g_j$ given by the formula (2.1).

(ii) $H^0(C) = \mathbb{C}$ is pure of type $(0, 0)$.

(iii) $H^2(C) = \mathbb{C}^r$ is pure of type $(1, 1)$.

(iv) The mixed Hodge numbers of the MHS on $H^1(C)$ are given by

$$h^{0,0}(H^1(C)) = \sum_{j,k} ((r(C_j, a_k^j) - 1) + \sum_{b \in B} (n(b) - 1) - r + 1,$$
In particular, one has the following formula for the first Betti number of \(C\).

\[
b_1(C) = \sum_{j,k} ((r(C_j, a^j_k) - 1) + \sum_{b \in B} (n(b) - 1) - r + 1 + 2 \sum_{j} g_j.
\]

Note that a point in the intersection \(A \cap B\) will give a contribution to the last two sums in the above formula for \(P(C)\).

**Example 2.3.** Suppose \(C\) is a nodal curve. Then for each singularity \(a^j_k \in A\) one has \(a^j_k \notin B\) (otherwise we get worse singularities than nodes) and \(r(a^j_k) = 2\). Moreover, each point \(b \in B\) satisfies \(n(b) = 2\). It follows that in this case we get

\[
P(C)(u, v) = ruv - \left( \sum_{j=1}^{r} g_j \right) u - \left( \sum_{j=1}^{r} g_j \right) v + r - n_2,
\]

with \(n_2\) the number of nodes of \(C\). More precisely, in this case we have \(n_2 = n_2' + n_2''\), where \(n_2'\) (resp. \(n_2''\)) is the number of nodes of \(C\) in \(A\) (resp. in \(B\)) and one clearly has

\[
n_2' = S_1 := \sum_{j,k} ((r(C_j, a^j_k) - 1), \quad n_2'' = S_2 := \sum_{b \in B} (n(b) - 1).
\]

**Example 2.4.** Suppose \(C\) has only nodes and ordinary triple points as singularities. Then let \(n_3\) be the number of triple points and note that we can write as above \(n_3 = n_3' + n_3''\), where \(n_3'\) (resp. \(n_3''\)) is the number of triple points of \(C\) in \(A_0 = A \setminus B\) (resp. in \(B\)). For a point \(a \in A_0\), the contribution to the sum \(S_1\) is 2, while the contribution to the sum \(S_2\) is 0.

A point \(b \in B\) can be of two types. The first type, corresponding to the partition \(3 = 1 + 1 + 1\), is when \(b\) is the intersection of three components \(C_j\), all smooth at \(b\). The contribution of such a point \(b\) is 0 to the sum \(S_1\) and 2 to the sum \(S_2\).

The second type, corresponding to the partition \(3 = 2+1\), is when \(b\) is the intersection of two components, say \(C_i\) and \(C_j\), such that \(C_i\) has a node at \(b\), and \(C_j\) is smooth at \(b\). The contribution of such a point \(b\) is 1 to the sum \(S_1\) and 1 to the sum \(S_2\).

It follows that the contribution of any triple point to the sum \(S_1 + S_2\) is equal to 2. Since the double points in \(C\) can be treated exactly as in Example 2.3, this yields the following.

\[
P(C)(u, v) = ruv - \left( \sum_{j=1}^{r} g_j \right) u - \left( \sum_{j=1}^{r} g_j \right) v + r - n_2 - 2n_3.
\]

When there are only triple points in \(B\) of the first type, then we obviously have the following additional relations

\[
S_1 = n_2' + 2n_3', \quad S_2 = n_2'' + 2n_3''.
\]

**Example 2.5.** Suppose \(C\) has only ordinary points of multiplicity 2, 3 and 4 as singularities. Then let \(n_4\) be the number of points of multiplicity 4 and note that we can write as above \(n_4 = n_4' + n_4''\), where \(n_4'\) (resp. \(n_4''\)) is the number of points of multiplicity
4 of $C$ in $A_0 = A \setminus B$ (resp. in $B$). For a point $a \in A_0$ of multiplicity 4, the contribution to the sum $S_1$ is 3, while the contribution to the sum $S_2$ is 0.

A point $b \in B$ can be of 4 types. The first type, corresponding to the partition $4 = 1 + 1 + 1 + 1$, is when $b$ is the intersection of 4 components $C_j$, all smooth at $b$. The contribution of such a point $b$ is 0 to the sum $S_1$ and 3 to the sum $S_2$.

The second type, corresponding to the partition $4 = 2 + 1 + 1$, is when $b$ is the intersection of 3 components, say $C_i, C_j$ and $C_k$, such that $C_i$ has a node at $b$, and $C_j$ and $C_k$ are smooth at $b$. The contribution of such a point $b$ is 1 to the sum $S_1$ and 2 to the sum $S_2$.

The third type, corresponding to the partition $4 = 2 + 2$, is when $b$ is the intersection of 2 components, say $C_i$ and $C_k$, such that $C_i$ and $C_k$ have a node at $b$. The contribution of such a point $b$ is 2 to the sum $S_1$ and 1 to the sum $S_2$.

The fourth type, corresponding to the partition $4 = 3 + 1$, is when $b$ is the intersection of 2 components, say $C_i$ and $C_k$, such that $C_i$ has a triple point at $b$, and $C_k$ is smooth at $b$. The contribution of such a point $b$ is 2 to the sum $S_1$ and 1 to the sum $S_2$.

It follows that the contribution of any point of multiplicity 4 to the sum $S_1 + S_2$ is equal to 3. Since the double and triple points in $C$ can be treated exactly as in Example 2.4, this yields the following.

$$P(C)(u, v) = ruv - \left( \sum_{j=1}^{r} g_j \right) u - \left( \sum_{j=1}^{r} g_j \right) v + r - n_2 - 2n_3 - 3n_4.$$ 

When there are only points of multiplicity 4 in $B$ of the first type, then we obviously have the following additional relations

$$S_1 = n_2' + 2n_3' + 3n_4'', \quad S_2 = n_2'' + 2n_3'' + 3n_4''.$$ 

Let’s look now at the cohomology of the smooth surface $U = \mathbb{P}^2 \setminus C$. By the additivity we get $P(U) = P(\mathbb{P}^2) - P(C)$ where $P(\mathbb{P}^2) = u^2v^2 + uv + 1$. This yields the following consequence.

**Corollary 2.6.**

$$P(U)(u, v) = u^2v^2 - (r - 1)uv + \left( \sum_{j=1}^{r} g_j \right) u + \left( \sum_{j=1}^{r} g_j \right) v - (r - 1) + 
+ \sum_{j,k} (r(C_j, a_k^j) - 1) + \sum_{b \in B} (n(b) - 1).$$

The contribution of $H^2_c(U, \mathbb{C})$ to $P(U)$ is the term $u^2v^2$, and that of $H^2_c(U, \mathbb{C})$ is the term $-(r - 1)uv$. Moreover, the dimension $\dim Gr^1_H^2(U, \mathbb{C})$ is the number of independent classes of type $(1,2)$, which correspond to classes of type $(1,0)$ in $H^2_c(U)$, and hence to the terms in $u$ in $P(U)$. For both statements see the proof of Theorem 2.1 in [1]. This proves the following result.

**Theorem 2.7.**

$$\dim Gr^1_H^2(U, \mathbb{C}) = \sum_{j=1}^{r} g_j$$
and
\[ \dim \text{Gr}^2 F H^2(U, \mathbb{C}) = \sum_{j=1}^{r} g_j + \sum_{j,k} ((r(C_j, a^j_k) - 1) + \sum_{b \in B} (n(b) - 1) - r + 1. \]

In particular, all the components \(C_j\) of the curve \(C\) are rational if and only if \(H^2(U)\) is pure of type \((2,2)\).

**Example 2.8.** Suppose \(C\) has only ordinary points of multiplicity 2, 3 and 4 as singularities. Then let \(n_k\) be the number of points of multiplicity \(k\), for \(k = 2, 3, 4\) Then using Example 2.5, we get the formula
\[ \dim \text{Gr}^2 F H^2(U, \mathbb{C}) = \sum_{j=1}^{r} g_j - r + 1 + n_2 + 2n_3 + 3n_4. \]

3. **Arrangements of transversely intersecting curves**

Recall that \(C = \bigcup_{j=1}^{r} C_j\) is the decomposition of \(C\) as a union of irreducible components \(C_j\), and the curve \(C_j\) has degree \(N_j\). In this section we assume that any curve \(C_j\) has only ordinary multiple points as singularities and let \(n_k(C_j)\) denote the number of ordinary points on \(C_j\) of multiplicity \(k\). We also assume that the intersection of any two distinct components \(C_i\) and \(C_j\) is transverse, i.e. the points in \(C_i \cap C_j\) are nodes of the curve \(C_i \cup C_j\). This implies in particular that \(A \cap B = \emptyset\). The formulas (2.1) and (2.2) yield the equality.

\[ g_j = \left( \frac{N_j - 1}{2} \right) \left( \frac{N_j - 2}{2} \right) - \frac{1}{2} \sum_{k} \left( \mu(C_j, a^j_k) + r(C_j, a^j_k) - 1 \right), \]

Using this, Theorem 2.7 gives the formula
\[ \dim \text{Gr}^2 F H^2(U, \mathbb{C}) = \sum_{j=1}^{r} \left( \frac{(N_j - 1)(N_j - 2)}{2} \right) - \frac{1}{2} \sum_{j,k} \left( \mu(C_j, a^j_k) - r(C_j, a^j_k) + 1 \right) + \sum_{b \in B} (n(b) - 1) - r + 1. \]

If \(a^j_k\) is an ordinary \(m\)-multiple point on the curve \(C_j\), one has \( \mu(C_j, a^j_k) = (m-1)^2 \) and hence
\[ \mu(C_j, a^j_k) - r(C_j, a^j_k) + 1 = (m-1)(m-2). \]

If we denote by \(n'_m\) (resp. \(n''_m\)) the number of \(m\)-multiple points of \(C\) coming from just one component \(C_j\) (resp. from the intersection of several components \(C_j\)), we see that we have
\[ \sum_{j,k} \left( \mu(C_j, a^j_k) - r(C_j, a^j_k) + 1 \right) = \sum_{m} (m-1)(m-2)n'_m. \]

This equality explains the contribution of the points in \(A\). Now let \(b \in B\) such that \(n(b) = m\). The number of such points is precisely \(n''_m\). It follows that
\[ \sum_{b \in B} (n(b) - 1) = \sum_{m} (m-1)n''_m. \]
Let $1 \leq i < j \leq r$ and consider the intersection $C_i \cap C_j$. It contains exactly $N_i N_j$ points, since $C_i$ and $C_j$ intersect transversely. The sum $S = \sum_{1 \leq i < j \leq r} N_i N_j$ represents the number of all such intersection points. Note that a point $b \in B$ is counted in this sum exactly $\binom{n(b)}{2}$ times. This yields the following formula

$$2S = \sum_m m(m-1)n''_m.$$ 

These formulas give the following result.

**Theorem 3.1.** With the above assumptions and notation, one has

$$\dim Gr^2_F H^2(U, \mathbb{C}) = \left( \begin{array}{c} N - 1 \\ 2 \end{array} \right) - \sum_m \left( \begin{array}{c} m - 1 \\ 2 \end{array} \right)n_m,$$

with $n_m = n'_m + n''_m$ the number of ordinary $m$-tuple points of $C$.

The following consequence of Theorem 2.7 and Theorem 3.1 applies in particular to any projective line arrangement.

**Corollary 3.2.** Assume that $C = \bigcup_{j=1}^r C_j$ is the decomposition of $C$ as a union of irreducible components $C_j$, with any curve $C_j$ having only ordinary multiple points as singularities and being rational, i.e. $g_j = 0$. If the intersection of any two distinct components $C_i$ and $C_j$ is transverse, i.e. the points in $C_i \cap C_j$ are nodes of the curve $C_i \cup C_j$, then one has

$$\dim H^2(U, \mathbb{C}) = \left( \begin{array}{c} N - 1 \\ 2 \end{array} \right) - \sum_m \left( \begin{array}{c} m - 1 \\ 2 \end{array} \right)n_m,$$

with $n_m$ the number of ordinary $m$-tuple points of $C$.

4. **Curves with ordinary singularities of multiplicity $\leq 4$**

Let $C \subset \mathbb{P}^2$ be a curve of degree $N$ having only ordinary singular points of multiplicity at most 4. Set $U = \mathbb{P}^2 \setminus C$, and let $C = \bigcup_{j=1}^r C_j$ be the decomposition of $C$ in irreducible components. Then,

$$P(C) = \sum_{j=1}^r P(C_j) - \sum_{0 \leq i < j \leq r} P(C_i \cap C_j) + \sum_{0 \leq i < j < k \leq r} P(C_i \cap C_j \cap C_k) - \sum_{0 \leq i < j < k < l \leq r} P(C_i \cap C_j \cap C_k \cap C_l).$$

Let $a^j_m$ denote the number of singular points of multiplicity $m$ that belong to the component $C_j$ (note that a point can be singular on two components, being a node on each of them).

Denote by $b^k_3$ (respectively $b^k_4$) the number of triple points (respectively points of multiplicity 4) of $C$ that are intersection of exactly $k$ components, for $k = 2, 3$ (respectively $k = 3, 4$). Let $b^2_4$ (respectively $\tilde{b}^2_4$) be the number of singular points $p$ of multiplicity 4 that

$$2S = \sum_m m(m-1)n''_m.$$ 

These formulas give the following result.

**Theorem 3.1.** With the above assumptions and notation, one has

$$\dim Gr^2_F H^2(U, \mathbb{C}) = \left( \begin{array}{c} N - 1 \\ 2 \end{array} \right) - \sum_m \left( \begin{array}{c} m - 1 \\ 2 \end{array} \right)n_m,$$

with $n_m = n'_m + n''_m$ the number of ordinary $m$-tuple points of $C$.

The following consequence of Theorem 2.7 and Theorem 3.1 applies in particular to any projective line arrangement.

**Corollary 3.2.** Assume that $C = \bigcup_{j=1}^r C_j$ is the decomposition of $C$ as a union of irreducible components $C_j$, with any curve $C_j$ having only ordinary multiple points as singularities and being rational, i.e. $g_j = 0$. If the intersection of any two distinct components $C_i$ and $C_j$ is transverse, i.e. the points in $C_i \cap C_j$ are nodes of the curve $C_i \cup C_j$, then one has

$$\dim H^2(U, \mathbb{C}) = \left( \begin{array}{c} N - 1 \\ 2 \end{array} \right) - \sum_m \left( \begin{array}{c} m - 1 \\ 2 \end{array} \right)n_m,$$

with $n_m$ the number of ordinary $m$-tuple points of $C$.

4. **Curves with ordinary singularities of multiplicity $\leq 4$**

Let $C \subset \mathbb{P}^2$ be a curve of degree $N$ having only ordinary singular points of multiplicity at most 4. Set $U = \mathbb{P}^2 \setminus C$, and let $C = \bigcup_{j=1}^r C_j$ be the decomposition of $C$ in irreducible components. Then,

$$P(C) = \sum_{j=1}^r P(C_j) - \sum_{0 \leq i < j \leq r} P(C_i \cap C_j) + \sum_{0 \leq i < j < k \leq r} P(C_i \cap C_j \cap C_k) - \sum_{0 \leq i < j < k < l \leq r} P(C_i \cap C_j \cap C_k \cap C_l).$$

Let $a^j_m$ denote the number of singular points of multiplicity $m$ that belong to the component $C_j$ (note that a point can be singular on two components, being a node on each of them).

Denote by $b^k_3$ (respectively $b^k_4$) the number of triple points (respectively points of multiplicity 4) of $C$ that are intersection of exactly $k$ components, for $k = 2, 3$ (respectively $k = 3, 4$). Let $b^2_4$ (respectively $\tilde{b}^2_4$) be the number of singular points $p$ of multiplicity 4 that

$$2S = \sum_m m(m-1)n''_m.$$ 

These formulas give the following result.

**Theorem 3.1.** With the above assumptions and notation, one has

$$\dim Gr^2_F H^2(U, \mathbb{C}) = \left( \begin{array}{c} N - 1 \\ 2 \end{array} \right) - \sum_m \left( \begin{array}{c} m - 1 \\ 2 \end{array} \right)n_m,$$

with $n_m = n'_m + n''_m$ the number of ordinary $m$-tuple points of $C$.

The following consequence of Theorem 2.7 and Theorem 3.1 applies in particular to any projective line arrangement.

**Corollary 3.2.** Assume that $C = \bigcup_{j=1}^r C_j$ is the decomposition of $C$ as a union of irreducible components $C_j$, with any curve $C_j$ having only ordinary multiple points as singularities and being rational, i.e. $g_j = 0$. If the intersection of any two distinct components $C_i$ and $C_j$ is transverse, i.e. the points in $C_i \cap C_j$ are nodes of the curve $C_i \cup C_j$, then one has

$$\dim H^2(U, \mathbb{C}) = \left( \begin{array}{c} N - 1 \\ 2 \end{array} \right) - \sum_m \left( \begin{array}{c} m - 1 \\ 2 \end{array} \right)n_m,$$

with $n_m$ the number of ordinary $m$-tuple points of $C$.
Indeed, a point of type $b_2$ (resp. $b_3$, resp. $b_4$) occurs only in one intersection $C_i \cap C_j$, and has the multiplicity 2 (resp.3, resp. 4) in this intersection. A point of type $a$ or $\sum$ represents the intersection of exactly 2 components, such that one of which has a triple point at $p$ (respectively each one has a node at $p$). Then one has

$$\sum_{0 \leq i < j \leq r} P(C_i \cap C_j) = \sum_{0 \leq i < j \leq r} N_i N_j - b_3^2 - 3b_4^2 - 2b_5^2 - 2b_6^2.$$

Indeed, a point of type $b_3^2$ (resp. $b_4^2$, resp. $b_5^2$) occurs only in one intersection $C_i \cap C_j$, and has the multiplicity 2 (resp.3, resp. 4) in this intersection. A point of type $b_4^3$ occurs in 3 intersections $C_i \cap C_j$ with multiplicities 1, 2, 2, and this accounts for the correction term $-2b_4^3$. Then one has

$$\sum_{0 \leq i < j < k \leq r} P(C_i \cap C_j \cap C_k) = b_3^3 + b_4^3 + \left(\frac{4}{3}\right)b_4^3,$$

and

$$\sum_{0 \leq i < j < k < l \leq r} P(C_i \cap C_j \cap C_k \cap C_l) = b_4^4.$$

Hence, by Proposition 2.1, we get the following.

$$P(C) = ruv - \left(\sum_{j=1}^{r} g_j u - \sum_{j=1}^{r} g_j v - \sum_{j=1}^{r} (a_2^j + 2a_3^j + 3a_4^j) - \sum N_i N_j \right)$$

$$+ b_3^2 + 3b_4^2 + 2b_5^2 + 3b_6^2 + b_7^3 + 3b_8^4.$$

Therefore, as above, we obtain

$$P(U) = u^2 v^2 - (r - 1)uv + 1 - r + \left(\sum_{j=1}^{r} g_j u + \sum_{j=1}^{r} g_j v + \sum_{j=1}^{r} (a_2^j + 3a_3^j + 6a_4^j) \right)$$

$$- \sum_{j=1}^{r} (a_3^j + 3a_4^j) + \sum N_i N_j - b_3^2 - 3b_4^2 - 2b_5^2 - 3b_6^2 - 3b_7^3 - 3b_8^4.$$

Finally we get

$$\dim Gr^2 P^2(U) = \sum_{j=1}^{r} (g_j + a_2^j + 3a_3^j + 6a_4^j - 1) + \sum N_i N_j + 1 - \sum_{j=1}^{r} (a_3^j + b_4^2 + b_5^2)$$

$$- 3\sum_{j=1}^{r} (a_4^j + \tilde{b}_4^2 + b_5^2 + b_6^4) + b_7^2$$

$$= \frac{(N - 1)(N - 2)}{2} - n_3 - 3n_4 + b_4^2,$$

with $n_m$ the number of ordinary $m$-tuple points of $C$.

**Theorem 4.1.** Let $C \subset \mathbb{P}^2$ be a curve of degree $N$ having only ordinary singular points of multiplicity at most 4. If $U = \mathbb{P}^2 \setminus C$, then one has

$$\dim Gr^2 H^2(U, \mathbb{C}) = \frac{(N - 1)(N - 2)}{2} - \sum_{m=3,4} \left(\frac{m - 1}{2}\right) n_m + b_4^2,$$
with \( n_m \) the number of ordinary \( m \)-tuple points of \( C \) and \( b_1^3 \) the number of singular points \( p \) of \( C \) which are smooth on one component \( C_i \) of \( C \) and have multiplicity 3 on the other component \( C_j \) of \( C \) passing through \( p \).

5. Pole order filtration versus Hodge filtration for plane curve complements

For any hypersurface \( V \) in a projective space \( \mathbb{P}^n \), the cohomology groups \( H^*(U, \mathbb{C}) \) of the complement \( U = \mathbb{P}^n \setminus V \) have a pole order filtration \( P^k \), see for instance [8], and it is known by the work of P. Deligne, A. Dimca [3] and M. Saito [11] that one has

\[
F^k H^m(U, \mathbb{C}) \subset P^k H^m(U, \mathbb{C})
\]

for any \( k \) and any \( m \). For \( m = 0 \) and \( m = 1 \), the above inclusions are in fact equalities (the case \( m = 0 \) is obvious and the case \( m = 1 \) follows from the equality \( F^1 H^1(U, \mathbb{C}) = H^1(U, \mathbb{C}) \)). For \( m = 2 \), we have again \( F^k H^2(U, \mathbb{C}) = P^k H^2(U, \mathbb{C}) \) for \( k = 0, 1 \) for obvious reasons, but one may get strict inclusions

\[
F^2 H^2(U, \mathbb{C}) \neq P^2 H^2(U, \mathbb{C})
\]

already in the case when \( V = C \) is a plane curve, see [5], Remark 2.5 or [4]. However, to give such examples of plane curves was until now rather complicated. We give below a numerical condition which tells us exactly when the above strict inclusion holds.

We need first to recall some basic definitions. Let \( S = \oplus_r S_r = \mathbb{C}[x, y, z] \) be the graded ring of polynomials with complex coefficients, where \( S_r \) is the vector space of homogeneous polynomials of \( S \) of degree \( r \). For a homogeneous polynomial \( f \) of degree \( N \), define the Jacobian ideal of \( f \) to be the ideal \( J_f \) generated in \( S \) by the partial derivatives \( f_x, f_y, f_z \) of \( f \) with respect to \( x, y \) and \( z \). The graded Milnor algebra of \( f \) is given by

\[
M(f) = \oplus_r M(f)_r = S/J_f.
\]

Note that the dimensions \( \dim M(f)_r \) can be easily computed in a given situation using some computer software e.g. Singular. Now we can state the main result of this section.

**Theorem 5.1.** Let \( C : f = 0 \) be a reduced curve of degree \( N \) in \( \mathbb{P}^2 \) having only weighted homogeneous singularities and let \( C_i \) for \( i = 1, ..., r \) be the irreducible components of \( C \). If \( U = \mathbb{P}^2 \setminus C \), then

\[
\dim P^2 H^2(U, \mathbb{C}) - \dim F^2 H^2(U, \mathbb{C}) = \tau(C) + \sum_{i=1}^{r} g_i - \dim M(f)_{2N-3},
\]

where \( \tau(C) \) is the global Tjurina number of \( C \) (that is the sum of the Tjurina numbers of all the singularities of \( C \)) and \( g_i \) is the genus of the normalization of \( C_i \) for \( i = 1, ..., r \).

In particular we get the following result, which yields in particular a new proof for Theorem 1.3 in [7].

**Corollary 5.2.** If a reduced plane curve has only nodes as singularities, then one has

\[
\dim M(f)_{2N-3} = \tau(C) + \sum_{i=1}^{r} g_i.
\]
Proof. Indeed, it is known that for a nodal curve one has the equality $F^2H^2(U, \mathbb{C}) = P^2H^2(U, \mathbb{C})$, see [2] or [11]. □

Note that we have the following obvious consequence of Theorem 2.7.

**Corollary 5.3.** For a reduced plane curve $C$ one has

$$\dim P^2H^2(U, \mathbb{C}) - \dim F^2H^2(U, \mathbb{C}) \leq \sum_{i=1}^{r} g_i.$$  

Proof. Indeed, Theorem 2.7 can be restated as

$$\dim H^2(U, \mathbb{C}) - \dim F^2H^2(U, \mathbb{C}) = \sum_{i=1}^{r} g_i,$$

in view of the equality $F^1H^2(U, \mathbb{C}) = H^2(U, \mathbb{C})$, see [4], proof of Corollary 1.32, page 185. □

**Remark 5.4.** If a reduced plane curve $C$ has only rational irreducible components, i.e. $g_i = 0$ for all $i$, then the above inequality implies $F^2H^2(U, \mathbb{C}) = P^2H^2(U, \mathbb{C})$. This result can be regarded as an improvement of a part of the Remark 2.5 in [5], where the result is claimed only for curves with nodes and cusps as singularities.

The above discussion implies also the following result, which can be regarded as a generalization of Theorem 4.1 (A) in [1].

**Corollary 5.5.** If a reduced plane curve $C : f = 0$ has only weighted homogeneous singularities, then one has

$$0 \leq \dim M(f)_{2N-3} - \tau(C) \leq \sum_{i=1}^{r} g_i.$$  

In particular, if in addition the curve $C$ has only rational irreducible components, then one has

$$\dim M(f)_{2N-3} = \tau(C).$$

Now we give the proof of Theorem 5.1. Corollary 1.3 in [8] implies that

$$\dim P^2H^2(U, \mathbb{C}) = \dim H^2(U, \mathbb{C}) + \tau(C) - \dim M(f)_{2N-3}.$$  

On the other hand, Theorem 2.7 and the fact $\dim F^1H^2(U, \mathbb{C}) = H^2(U, \mathbb{C})$ yield

$$\dim F^2H^2(U, \mathbb{C}) = \dim H^2(U, \mathbb{C}) - \sum_{i=1}^{r} g_i,$$

which clearly completes the proof of Theorem 5.1.

**Example 5.6.** In this example we present a free divisor $C : f = 0$, whose irreducible components consist of 12 lines and one elliptic curve, and where $F^2H^2(U, \mathbb{C}) \neq P^2H^2(U, \mathbb{C})$. Let $f = xyz(x^3 + y^3 + z^3)[(x^3 + y^3 + z^3)^3 - 27x^3y^3z^3]$. If we consider the pencil of cubic curves $(x^3 + y^3 + z^3, xyz)$, then the curve $C$ contains all the singular fibers of this pencil, and this accounts for the 12 lines given by

$$xyz[(x^3 + y^3 + z^3)^3 - 27x^3y^3z^3] = 0,$$
and the elliptic curve (hence of genus 1) given by $x^3 + y^3 + z^3 = 0$. Then $C$ is a free divisor, see [13] or by a direct computation using Singular, which shows that $I = J_f$, where $I$ is the saturation of the Jacobian ideal $J_f$, see Remark 4.7 in [6]. The direct computation by Singular also yields $\tau(C) = 156$ and $\dim M(f)_{2N-3} = \dim M(f)_{27} = 156$. Moreover, applying Corollary 1.5 in [9], we see via a Singular computation that all singularities of the curve $C$ are weighted homogeneous. Alternatively, there are 12 nodes, 3 in each of the 4 singular fibers of the pencils (which are triangles), and the 9 base points of the pencil, each an ordinary point of multiplicity 5. Each of the 12 lines contains exactly 3 of these base points, and they are exactly the intersection of the elliptic curve with the line. This description implies that there are no other singularities, in accord with

$$12 + 9 \times 16 = 156 = \tau(C).$$

It follows from Theorem 5.1 that $\dim P^2H^2(U, \mathbb{C}) - \dim F^2H^2(U, \mathbb{C}) = 1$. Hence the presence of a single irrational component of $C$ leads to $F^2H^2(U, \mathbb{C}) \neq P^2H^2(U, \mathbb{C})$.

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References


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