Numerical schemes for the simulation of seismic wave propagation in frequency domain

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\textsuperscript{3} TOTAL Exploration-Production
Motivation

Examples of seismic applications
Motivation

Imaging method: the full wave inversion

- Iterative procedure using the wavefield in order to obtain quantitative high resolution images of the subsurface physical parameters
Motivation

Imaging method: the full wave inversion

- Iterative procedure using the wavefield in order to obtain quantitative high resolution images of the subsurface physical parameters

Seismic imaging: time-domain or harmonic-domain?

- Time-domain: imaging condition complicated but low computational cost
- Harmonic-domain: imaging condition simple but huge computational cost
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Imaging method: the full wave inversion

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Seismic imaging: time-domain or harmonic-domain?

- Time-domain: imaging condition complicated but low computational cost
- Harmonic-domain: imaging condition simple but huge computational cost

Forward problem of the inversion process

- Elastic wave propagation in harmonic domain: Helmholtz equation
- Reduction of the size of the linear system
Motivation

Seismic imaging in heterogeneous complex media

- Complex topography
- High heterogeneities
Motivation

Seismic imaging in heterogeneous complex media

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Use of unstructured meshes with FE methods
Motivation

Seismic imaging in heterogeneous complex media

▶ Complex topography
▶ High heterogeneities

Use of unstructured meshes with FE methods

DG method

▶ Flexible choice of interpolation orders ($p - adaptativity$)
▶ Highly parallelizable method
▶ Increased computational cost as compared to classical FEM
Motivation

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- Complex topography
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DG method

- Flexible choice of interpolation orders ($p$ – adaptativity)
- Highly parallelizable method
- Increased computational cost as compared to classical FEM
Motivation

Objective of this work

- Development of an hybridizable DG (HDG) method
- Comparison with a reference method: a standard nodal DG method

**Figure**: Degrees of freedom of DGM

**Figure**: Degrees of freedom of HDGM
HDG methods

HDG methods


Contents

2D Helmholtz elastic equations

Notations and definitions

Hybridizable Discontinuous Galerkin method

Numerical results

Conclusions-Perspectives
2D Helmholtz elastic equations

First order formulation of Helmholtz wave equations

\[ x = (x, y) \in \Omega \subset \mathbb{R}^2, \]

\[
\begin{align*}
  i\omega \rho(x)v(x) &= \nabla \cdot \sigma(x) + f_s(x) \\
  i\omega \sigma(x) &= \mathcal{C}(x) \varepsilon(v(x))
\end{align*}
\]

- Free surface condition: \( \sigma n = 0 \) on \( \Gamma_l \)
- Absorbing boundary condition: \( \sigma n = v_p(v \cdot n)n + v_s(v \cdot t)t \) on \( \Gamma_a \)

- \( v \): velocity vector
- \( \sigma \): stress tensor
- \( \varepsilon \): strain tensor
2D Helmholtz elastic equations

First order formulation of Helmholtz wave equations
\[ \mathbf{x} = (x, y) \in \Omega \subset \mathbb{R}^2, \]
\[
\begin{align*}
  i\omega \rho(x)\mathbf{v}(x) &= \nabla \cdot \mathbf{\sigma}(x) + f_s(x) \\
  i\omega \mathbf{\sigma}(x) &= C(x) \varepsilon(\mathbf{v}(x))
\end{align*}
\]

- Free surface condition: \( \mathbf{\sigma} \mathbf{n} = 0 \) on \( \Gamma_l \)
- Absorbing boundary condition: \( \mathbf{\sigma} \mathbf{n} = v_p (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} + v_s (\mathbf{v} \cdot \mathbf{t}) \mathbf{t} \) on \( \Gamma_a \)

- \( \rho \): mass density
- \( C \): tensor of elasticity coefficients
- \( v_p \): P-wave velocity
- \( v_s \): S-wave velocity
- \( f_s \): source term, \( f_s \in L^2(\Omega) \)
Contents

2D Helmholtz elastic equations

Notations and definitions

Hybridizable Discontinuous Galerkin method

Numerical results

Conclusions-Perspectives
Notations and definitions

Notations

- $\mathcal{T}_h$ mesh of $\Omega$ composed of triangles $K$
Notations and definitions

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- $\mathcal{T}_h$ mesh of $\Omega$ composed of triangles $K$
- $\mathcal{F}_h$ set of all faces $F$ of $\mathcal{T}_h$
Notations and definitions

Notations

- $\mathcal{T}_h$ mesh of $\Omega$ composed of triangles $K$
- $\mathcal{F}_h$ set of all faces $F$ of $\mathcal{T}_h$
- $\mathbf{n}$ the normal outward vector of an element $K$
Notations and definitions

Definitions

- Jump $[\cdot]$ of a vector $v$ through $F$:
  
  $[v] = v^+ \cdot n^+ + v^- \cdot n^- = v^+ \cdot n^+ - v^- \cdot n^-$

- Jump of a tensor $\sigma$ through $F$:
  
  $[\sigma] = \sigma^+ n^+ + \sigma^- n^- = \sigma^+ n^+ - \sigma^- n^+$
Contents

2D Helmholtz elastic equations

Notations and definitions

Hybridizable Discontinuous Galerkin method

Formulation

Numerical results

Conclusions-Perspectives
HDG formulation of the equations

Local HDG formulation

\[
\begin{align*}
\int_K i \omega \rho^K v^K \cdot w + \int_K \sigma^K : \nabla w - \int_{\partial K} \hat{\sigma} \partial K \cdot n \cdot w &= 0 \\
\int_K i \omega \sigma^K : \xi + \int_K v^K \cdot \nabla \cdot (C^K \xi) - \int_{\partial K} \hat{v} \partial K \cdot C^K \xi \cdot n &= 0
\end{align*}
\]
HDG formulation of the equations

Local HDG formulation

\[
\begin{aligned}
\int_{K} i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_{K} \sigma^K : \nabla \mathbf{w} - \int_{\partial K} \hat{\sigma}^{\partial K} \cdot \mathbf{n} \cdot \mathbf{w} &= 0 \\
\int_{K} i\omega \sigma^K : \xi + \int_{K} \mathbf{v}^K \cdot \nabla \cdot \left( C^K \xi \right) - \int_{\partial K} \hat{\mathbf{v}}^{\partial K} \cdot C^K \xi \cdot \mathbf{n} &= 0
\end{aligned}
\]

We define:

\[
\begin{aligned}
\hat{\mathbf{v}}^F &= \lambda^F, & \forall F \in \mathcal{F}_h, \\
\hat{\sigma}^{\partial K} \cdot \mathbf{n} &= \sigma^K \cdot \mathbf{n} - \tau \mathbf{l} \left( \mathbf{v}^K - \lambda^{\partial K} \right), & \text{on } \partial K
\end{aligned}
\]

where \( \tau \) is the stabilization parameter (\( \tau > 0 \))

\( \hat{\sigma}^K \) and \( \hat{\mathbf{v}}^K \) are numerical traces of \( \sigma^K \) and \( \mathbf{v}^K \) respectively on \( \partial K \)
HDG formulation of the equations

Local HDG formulation

We replace $\hat{\mathbf{v}}^K$ and $(\hat{\sigma}^K \cdot \mathbf{n})$ by their definitions into the local equations:

$$
\begin{aligned}
\int_K i \omega \rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \sigma^K : \nabla \mathbf{w} - \int_{\partial K} \sigma^K \cdot \mathbf{n} \cdot \mathbf{w} \\
+ \int_{\partial K} \tau \mathbf{I} \left( \mathbf{v}^K - \lambda \partial K \right) \cdot \mathbf{w} = 0
\end{aligned}
\quad
\begin{aligned}
\int_K i \omega \sigma^K : \xi + \int_K \mathbf{v}^K \cdot \nabla \cdot \left( \mathbf{C}^K \xi \right) - \int_{\partial K} \lambda \partial K \cdot \mathbf{C}^K \xi \cdot \mathbf{n} = 0
\end{aligned}
$$
Local HDG formulation

\[
\begin{align*}
&\int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K \left( \nabla \cdot \sigma^K \right) \cdot \mathbf{w} + \int_{\partial K} \tau I \left( \mathbf{v}^K - \lambda \partial K \right) \cdot \mathbf{w} = 0 \\
&\int_K i\omega \sigma^K : \mathbf{\xi} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left( \mathbf{C}^K \mathbf{\xi} \right) - \int_{\partial K} \lambda \partial K \cdot \mathbf{C}^K \mathbf{\xi} \cdot \mathbf{n} = 0
\end{align*}
\]
HDG formulation of the equations

Transmission condition

In order to determine $\lambda^K$, the continuity of the normal component of $\hat{\sigma}^K$ is weakly enforced, rendering this numerical trace conservative:

$$
\int_F \left[ \hat{\sigma}^K \cdot \mathbf{n} \right] \cdot \eta = 0
$$
Transmission condition

In order to determine $\lambda^K$, the continuity of the normal component of $\hat{\sigma}^K$ is weakly enforced, rendering this numerical trace conservative:

$$\int_F [\hat{\sigma}^K \cdot n] \cdot \eta = 0$$

Replacing $(\hat{\sigma}^K \cdot n)$ and summing over all faces, the transmission condition becomes:

$$\sum_{K \in T_h} \int_{\partial K} (\sigma^K \cdot n) \cdot \eta - \sum_{K \in T_h} \int_{\partial K} \tau I (v^K - \lambda \partial K) \cdot \eta = 0$$
HDG formulation of the equations

Global HDG formulation

\[
\begin{align*}
\int_K i\omega \rho^K v^K \cdot w - \int_K (\nabla \cdot \sigma^K) \cdot w + \int_{\partial K} \tau l (v^K - \lambda \partial^K) \cdot w &= 0 \\
\int_K i\omega \sigma^K : \xi + \int_K v^K \cdot \nabla \cdot \left( \frac{C^K \xi}{\xi} \right) - \int_{\partial K} \lambda \partial^K \cdot \frac{C^K \xi}{\xi} \cdot n &= 0 \\
\sum_{K \in T_h} \int_{\partial K} (\sigma^K \cdot n) \cdot \eta - \sum_{K \in T_h} \int_{\partial K} \tau l (v^K - \lambda \partial^K) \cdot \eta &= 0
\end{align*}
\]
Numerical results

Contents

2D Helmholtz elastic equations

Notations and definitions

Hybridizable Discontinuous Galerkin method

Numerical results
  Disk-shaped scatterer problem
  Marmousi test-case

Conclusions-Perspectives
Disk-shaped scatterer problem

- $a = 2000.0 m$ and $b = 8000.0 m$
- Physical parameters in $\Omega$:
  - $\rho = 1 kg.m^{-3}$
  - $\lambda = 8 GPa$
  - $\mu = 4 GPa$
- $\Gamma_l$ free surface boundary:
  $\sigma n = 0$
- $\Gamma_a$ absorbing boundary:
  $\sigma n = v_p (v \cdot n) n + v_s (v \cdot t)t$
- Three meshes:
  - 1200 elements
  - 5400 elements
  - 22000 elements

Computational domain $\Omega$ setting
## Disk-shaped scatterer problem

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Marmousi test-case

Computational domain $\Omega$ composed of 235000 triangles
Parallel results for the Marmousi test-case with the HDG-P2 scheme

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</table>
Contents

2D Helmholtz elastic equations

Notations and definitions

Hybridizable Discontinuous Galerkin method

Numerical results

Conclusions-Perspectives
Conclusions

- The HDG scheme has the correct convergence order \((p + 1)\)
- On a same mesh the HDG formulation is more competitive in terms of memory and computational time than the upwind flux DG formulation and the IPDG method
Conclusions

The HDG scheme has the correct convergence order \((p + 1)\)

On a same mesh the HDG formulation is more competitive in terms of memory and computational time than the upwind flux DG formulation and the IPDG method.

Perspectives

Develop 3D Upwind flux DG and HDG formulations for Helmholtz equations

Solution strategy for the HDG linear system
Thank you!